Educate~ Vol.7, No.2, 2007, pp. 30-45

Research Paper

The nature of mathematical enrichment: a case study of implementation

by Jennifer Piggott (jsp38@cam.ac.uk)
University of Cambridge, Faculty of Education.

Contextualization

I joined the NRICH mathematics project (www.nrich.maths.org) as the project director in late 2001, just as an internal evaluation was being completed. The evidence from this and previous evaluations and discussions with colleagues identified a need to review the project's work. Over the following months a series of meetings focussing on the purposes and the role of the project were held and resulted in three main outcomes. First, a set of revised aims were written. Second, in describing the project's activities and resources, terms such as "enrichment" and "problem solving" were being used without any clarity about what was meant; this needed to be addressed. Third, there was a desire to make the site and its contents more accessible and inclusive through the labelling and grouping of resources. One particular approach to addressing this last issue was the production of enrichment trails (ordered groups) of problems on particular problem solving or other mathematical themes. It was these trails and the desire to unpick what we meant by "enrichment" that informed the focus of the research.

Abstract. *This paper reports a framework for describing the nature of mathematics enrichment that emerged from a case study based on the work of the NRICH Project (www.nrich.maths.org) team when producing "mathematics enrichment trails" (an ordered set of related mathematics problems and support materials). A range of data sources, including the trails, trail development sessions, related literature and the views of colleagues were used to inform the findings. The data were analysed using NVivo and involved the development of two complementary coding systems. One, drawn from the data itself, gave evidence of views of the content aspects of mathematical enrichment. The other, specifically designed and informed by the literature, was used to aid the analysis of the roles of teaching and learning inherent in views of enrichment described by participants. The framework describes the content of an enrichment curriculum as well as implications for teaching and learning, the experiences of learners and the features of settings where this occurs. To support this, some detail is provided on the role, nature and purpose of problem-solving and what constitutes a good problem. While emerging from a particular context, the framework highlights the need for debate concerning the audience for mathematics enrichment, particularly in questioning the commonly held belief that its value is in supporting the needs of the mathematically most able. The framework also has potential value through offering a focus for debate within the wider community concerning the nature of mathematics enrichment and as a reference point for evaluating the potential of existing or new curriculum to deliver mathematics enrichment.*

Introduction

The study took place over a period of approximately two years (2003-2005). During this time a range of data was collected; this included interviews with colleagues on the NRICH team as they worked on the production of trails and with teachers using material from the trails, email correspondence, and the trails themselves. Throughout the study there was a continuous search for literature sources to help clarify issues or shed additional light on ideas that emerged. As a result the study involved an iterative process of data collection, literature review and revision of the framework.

Based on the existing experiences of the project team, the study started with a working definition of enrichment whose purpose was to support identifying a methodology, data sources and analytical tools that could illustrate, extend or refute that starting point.

The working definition of enrichment was described as something that affects learners' classroom experiences in a way that places greater emphasis on youngsters working together at being mathematical. The aim of enrichment is that learners will be more motivated by the subject and confident to use and apply their knowledge.

Views of Enrichment Found Within the Literature

The use of the term enrichment has undergone little rigorous examination and is almost exclusively in the context of provision for the most able (Martinson, 1968; Renzulli, 1977; Stanley, 1979; Worcester, 1979; UK-Mathematics-Foundation, 2000) as if it is reserved for the few:

> "Enrichment as a way of giving better educational opportunities to the mentally advanced child…"
>
> (Worcester, 1979, p. 98)

This does not exclude the possibility that enrichment can have a wider audience, simply that it is often not considered in this way (there are some notable exceptions e.g. Wallace, 1986). Some authors, including Stanley (1979) and Eyre and Marjoram (1990) view enrichment as something that is done in addition to the normal curriculum, such as visits to museums and participation in clubs and master classes. Clendening and Davis (1983) and Sheffield (1999) make some attempt to describe "enrichment" in terms of "depth, breadth and relevance". The terms "depth" and "breadth" are used loosely by the authors to refer to learners' level of understanding, while they refer to "relevance" in terms of the individual rather than some arbitrary generic definition. Clendening and Davis also imply a relationship between enrichment and the "normal" classroom in contrast to the view of Eyre and Marjoram. Others conceptualise enrichment as 'acceleration' (Stanley, 1979; Gross, 1999). Curricular and practical implementations of provision for able learners are also described as "extension programmes", of which there are a large number of examples internationally. Such examples include "Primary extension and Challenge" in Western Australia, provision for able learners in Gwinnett County in the US and summer schools offered by the National Academy for Gifted and Talented Youth in the UK. In particular, in the UK, in June 2004, the Department for Education and Skills (DfES) included in their response to the Post 14 Mathematics Inquiry (Smith, 2004) the intention to develop an "extension curriculum":

> "… we have asked QCA to develop guidance for an extension curriculum separately at KS3 and KS4 recognising that student engagement will be key."
>
> (DfES, 2004, p. 41)

Though, in this case, as with many of the other examples, what is meant by "extension" is not clearly defined.

Another view is offered by Keating (1979), who refers to enrichment as an administrative label (evident in Excellence in Cities Schools in England where there is a requirement to produce a list of gifted and talented learners (DfES, 2005). In a paper presented at the Cambridge Symposium on Education Research, Feng (2005) concluded that enrichment is currently poorly defined. The aim here is to move away from this "aura of vagueness and confusion" (Barbe 1960) and offer a definition of enrichment that not only supports the work of the NRICH Project but which can act as a focus for wider discussion and/or as an analytical tool in identifying what might be enrichment activities.

The two terms “acceleration” and “extension” are often used in the context of enrichment and therefore require some clarification. In this paper I am defining acceleration to be the intentional exposure of learners to more advanced standard curriculum subject matter with the specific aim of examination on that material in advance of chronological age. Renzulli (in George *et al*, 1979) does not dismiss acceleration as a way of meeting the needs of able learners but does raise the concern that it does not represent a radical departure from an able youngster’s usual experiences:

> “acceleration is basically a means for quantitative rather than qualitative differentiation” (p. 190).

In this sense acceleration is only of significant value to the very few students for whom more of the same and faster is appropriate.

Extension is considered to be the exposure of learners to content not normally found in the standard curriculum and which might be considered appropriate to that chronological age or older, including: the opportunity to learn new mathematical content or techniques (such as an introduction to group theory), application of an area of mathematics to different contexts not normally covered within the curriculum (such as some applications to art or astronomy), and the study of mathematics as a cultural, social or historical phenomenon. Extension therefore includes the opportunity to learn more mathematics. This can be enriching if it arises naturally out of situations and is developed through interest and need rather than seen as a requirement.

The working definition of enrichment given earlier includes the phrase “being mathematical”, which can be taken to include the application of mathematics to a range of contexts, and the engagement in learning about mathematics as a social phenomenon (both described above as extension). These opportunities can occur through direct experience of applying knowledge to novel contexts or through viewing mathematics as developed, seen and used by others (for example observing someone else “being mathematical” by reading an article), or as reflected in the world around them. In terms therefore of its early working definition, enrichment could be seen to encompass extension, whereas doing more of the same, as implied by acceleration, does not.

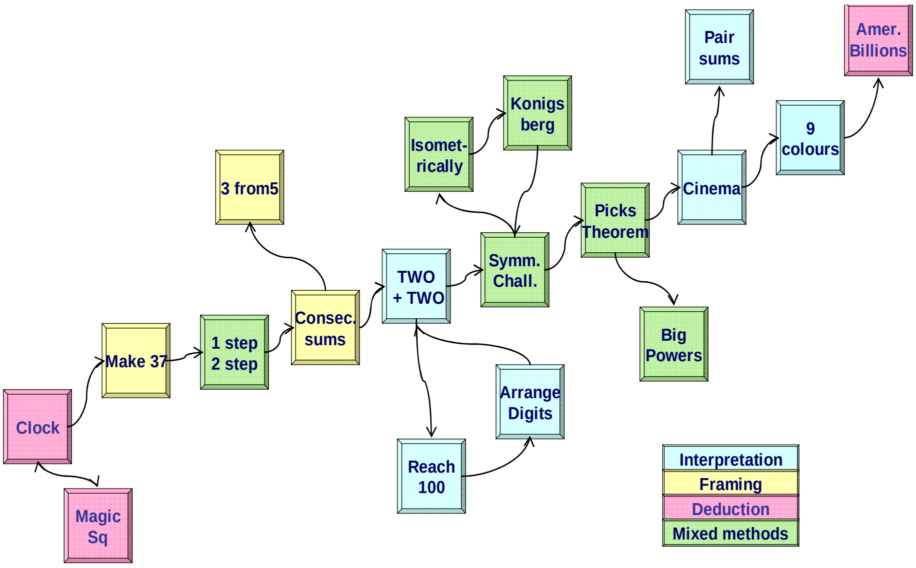
Methodology

As the NRICH project director, I have responsibility for overall strategy and long term planning, and the development of the NRICH site and other artefacts, such as trails and activities (such as professional development courses). I also have a monitoring role that, as I engaged in the research, became part of the process of data-collection and feedback. It was in this latter area that the tension between researcher and director was cause for most concern. However, the role of director proved to be relatively separate from the work on trails and enrichment and my role as a participant researcher, reflecting on, while being party to, the practice, gave richness to the process and the outcome. The production of trails and the discussions with colleagues became part of an interactive research project.

My dual roles, rather than weakening the value of the project, strengthened the outcomes by enabling multiple iterations and review. Both roles complemented and supported each other. As a researcher, my role was to identify and analyse information gleaned from a range of sources in order to make sense of the team’s understanding of enrichment and related concepts. As a participant, I was echoing ideas or adding ideas into the trail development process, as colleagues worked on them and, in turn, revised their own views. There may be blurring in the originality of ideas but not in the ownership. Members of the team were confident and independent enough to say what they believed.

The research involved identifying and articulating findings within a very specific context, focussed on an area of immediate interest and need for the NRICH project, the nature of enrichment.

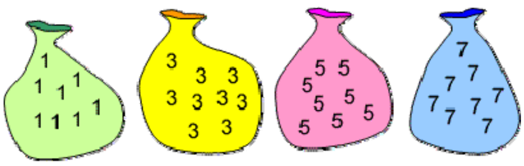
The participants involved in the study comprised five members of the NRICH team, a professional colleague and four teachers who used the trail materials. In addition, other members of the team and professional colleagues contributed to seminar and colloquia discussions and team meetings. The study aimed to make some sense of what might constitute common or contrasting sets of views. I was attempting to construct some meaning that might be described as shared by the participants but was actually, in the end, my understanding and not theirs. The outcomes had the potential to be of value to the wider community being based in practice, though not assuming an absolutist view of enrichment.
Data from the ten participants in the study came from seven sources: six interviews with members of the team, six extracts from team meetings, ten emails, three sets of transcripts of trail development sessions involving two team members and four trails (Being systematic, Generalising, Logo and Areas of Triangles). Versions of the first two trails have now been published in paper form (Piggott and Pumfrey, 2005; Piggott and Pumfrey, 2006) and an extract comprising the trail map and two problems from "Being Systematic" is given in Figure 1, four sets of notes from seminars and discussion forums, and five interviews with teachers using trail materials.



Make 37

Four bags contain a large number of 1s, 3s, 5s and 7s.

Pick any ten numbers from the bags above so that their total is 37.



1 Step 2 Step
Liam's house has a staircase with 12 steps. He can go down the steps one at a time or two at time.
For example: He could go down 1 step, then one step, then 2 steps, then 2, 2, 1, 1, 1, 1.
In how many different ways can Liam go down the 12 steps, taking one or two steps at a time?

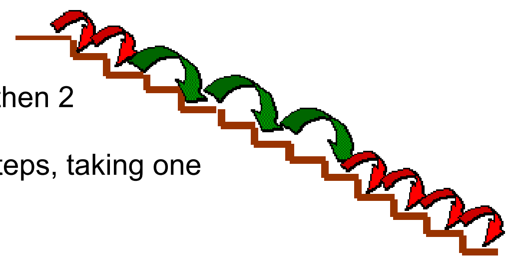


Figure 1. Two problems from the "Being Systematic" trail

All documentation (apart from the trails) was transcribed and entered into Nudist NVivo. The coding and analysis occurred in two phases. In the first phase, the coding framework (see Figure 2) was devised through an iterative process from some tentative starting points and through engagement with the data itself.

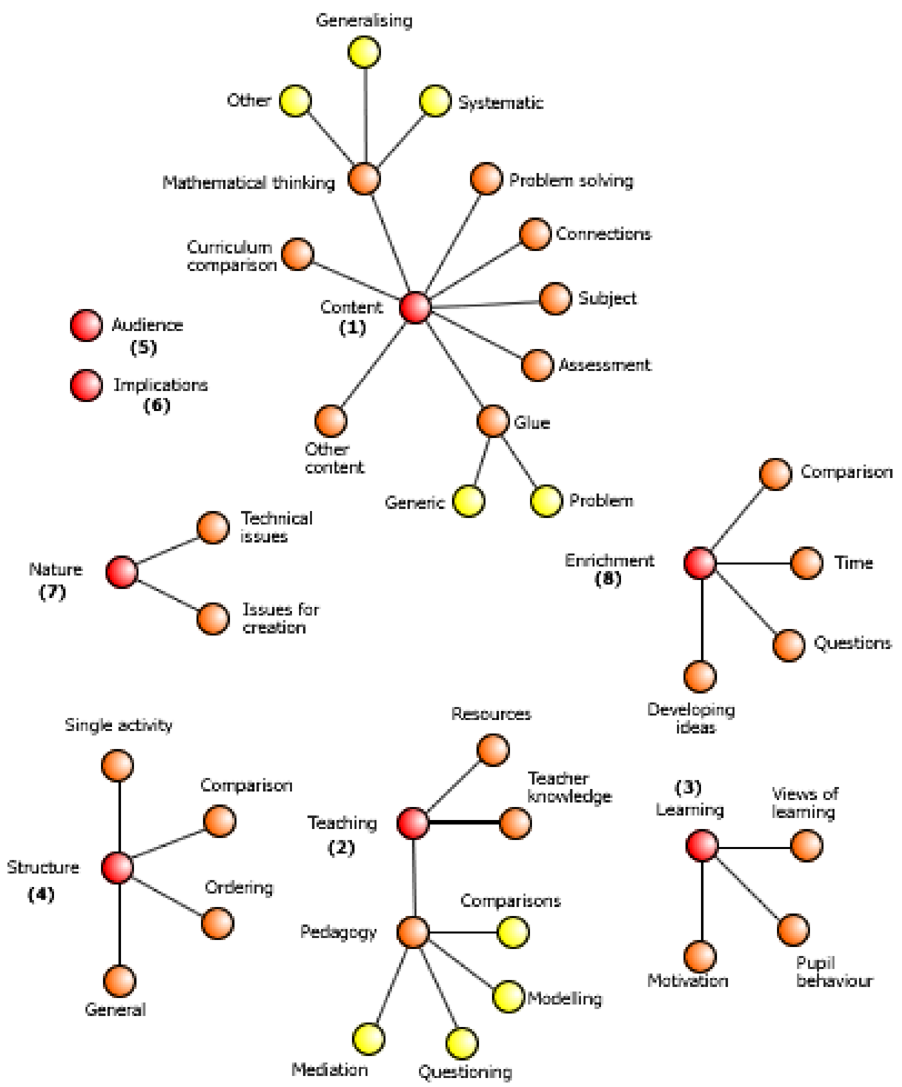


Figure 2. Nodes used in first coding

The final coding structure in this first phase reflected literature in the areas of enrichment, problem-solving and mathematical thinking, teaching and learning, my own preconceptions and ideas emerging from the data itself.

In the first phase of analysis, frequency counts on the number of passages coded under each node yielded information on the main features of what participants identified as elements of enrichment such as aspects of teaching and learning, including problem-solving and mathematical thinking, roles of teachers and learners, and specific examples of "good problems".

The data were then re-analysed by cross-referencing nodes. For example, how many times a passage was coded as being about problem-solving was also coded as an example of enrichment (and vice versa) is shown in the Table 1. The table illustrates that 163 passages were coded as referencing content and 113 teaching, with 29 of those passages common to both.

Table 1. Raw totals of cross-referenced passages.

	1 Content	2 Teaching	3 Learning	4 Structure	5 Audience	6 Implications	7 Nature	8 Enrichment	
1	48	29	25	23	1	0	2	35	163
2	29	17	35	6	3	2	2	19	113
3	25	35	17	5	0	0	3	2	87
4	23	6	5	1	0	0	3	1	39
5	1	3	0	0	0	0	2	1	7
6	0	2	0	0	0	0	2	1	5
7	2	2	3	3	2	2	1	1	16
8	35	19	2	1	1	1	1	3	63
	163	113	87	39	7	5	16	63	

This process revealed some strong links between particular features, with enrichment relating most strongly to content and teaching. Table 2 shows cross-references as percentages. Nodes which had less than 20 passages coded (nodes 5, 6, 7) have been omitted. The lack of symmetry in this table results from the calculation of the percentages based on the total passages in each node. In addition, references within a top-level node to other sibling nodes were included so, for example, 29 per cent (48) of the passages under "Content" (node 1) and its siblings reference other "Content nodes" and 18 per cent (29) reference nodes related to "teaching" (node 2) and its siblings;

Of initial concern was the apparent low correlation between passages coded as "learning and enrichment", but this link could be interpreted as existing through "teaching" as the bridging theme. This is because "enrichment" has a strong two-way correlation with "teaching" and with "learning" (each having the corresponding node as its highest percentage of links 31% and 40% respectively), with teaching appearing in 56 per cent of the passages coded for enrichment. This indirect link through teaching could be ascribed to the nature of the data sources (views of teachers and colleagues) that did not link directly to learners' experiences.

Table 2. Percentages of passages coded under the five major nodes

Nodes	1 Content	2 Teaching	3 Learning	4 Structure	8 Enrichment
1	29	26	29	59	56
2	18	15	40	15	30
3	15	31	20	13	3
4	14	5	6	3	2
8	21	17	2	3	5
	100	100	100	100	100

During the first phase of analysis, two main threads arose as needing more detailed investigation: first, the high profile of problem-solving as a core aspect of enrichment and second, the roles of teaching and learning. The former was examined in greater detail using the existing data analysis tools and further evidence from literature. However, it became evident that the coding system was not able to support the identification of more detailed features of teaching and learning which appeared to exist within the data. As a result, a second coding was undertaken using a structure based mainly on two sources (Greeno *et al*, 1996 and Ernest, 1991), plus the inclusion of references to the purposes of problem-solving identified by Stanic and Kilpatrick (1988) and Wilson *et al* (1993). The resulting analysis framework was categorised into three main perspectives: traditionalist, reflexive and pragmatic. Each of these perspectives were described in terms of the views of knowledge, learning and related teaching styles, and content types with which they might be associated and are listed in Table 3. It was the descriptors that were at the heart of the analysis and the categorisation of those descriptors under the three main headings (acting only as place-holders) added nothing to the process. Unlike the first phase of the coding, the second structure was imposed on the data, rather that being created out of the data. The coding structure and the number of passages coded within each category are also given in Table 3.

Table 3. Second coding related to teaching and learning

Perspectives

	Traditionalist		**Reflexive**		**Pragmatic**	
Aspects of teaching, learning and environmental factors	Stimulus-response	0	Use of misconceptions as a tool	1	Learners given opportunities to observe and practice activities	1
Aspects of teaching, learning and environmental factors	Clear rules and goals with feedback linked directly to those goals.	1	Social processes of understanding, sharing and explaining	8	Increasing skills and knowledge enabling engagement	3
Aspects of teaching, learning and environmental factors	Teachers act as experts giving knowledge to learners.	0	Use of metacognition by teacher and learner	4	Encourages a learner's personal identity as having individual knowledge that contributes to general understanding	5
Aspects of teaching, learning and environmental factors	Learners know what to expect and behave as passive receivers of knowledge	0	Encouraging the identification of unifying principles (generalising and connecting)	27	Development of practices of enquiry	5
Aspects of teaching, learning and environmental factors	Clearly defined rules and rote learning valued	0			Aspects of communal (and individual) sense making, enquiry and learning	9
Aspects of teaching, learning and environmental factors	Finely structured chunking of materials	5	Room to explore and manoeuvre.	33	Learning guided and supervised by masters	4
Aspects of teaching, learning and environmental factors			Different routes to solution valued	6	Conversations and social interactions within the community	3
Aspects of teaching, learning and environmental factors			Problem-solving (for)	7		
Content aspects	Ordered materials developing gradients of similarity to aid associations	11	Encourages learners to move from within understanding to the extension of understanding.	20	Supports progress in socio-cultural practices ’ cooperation, reasoning communication	3
Content aspects	Simple to complex where complex can be clearly linked (no surprises)	5	Learning through problem-solving, fostering understanding of concepts through exploration and investigation	12	Highlights the general that might be transferable	0
Content aspects	About problem-solving	1		19	"Meaningful" problematic situations	4

An Enrichment Framework

The study revealed a far too simplistic view of enrichment encapsulated in the original working definition and that a more complex model involving content, teaching, learning and impact was being described by participants. To reflect this finding, an enrichment framework was constructed. The resulting framework highlights the need for debate concerning the audience for mathematics enrichment, particularly in questioning the commonly held belief that its value is in supporting the needs of the mathematically most able. It also has potential value through offering a focus for debate within the wider community concerning the nature of mathematics enrichment and as a reference point for evaluating the potential of existing or new curriculum to deliver mathematics enrichment.

The framework is described below in terms of the four elements: content, experiences for learners, implications for teaching and its longer term potential influence on learners. It is followed by further discussion of some of its key features. It is not suggested that all mathematical enrichment opportunities should encompass every element described, but rather that the framework can act as a reference point for evaluating activities described as "enriching".

Aspects of enrichment associated with content

Enrichment involves offering learners opportunities to pose as well as solve challenging (non-routine) problems that allow for different methods, require fluency in the problem-solving processes and encourage the identification of elegant or efficient solutions (for problem-solving). Such problems might also broaden students' problem-solving skills (about problem-solving), deepen and broaden mathematical content knowledge such as revealing patterns, leading to generalisations or unexpected results (through problem-solving) and have potential to reveal underlying principles or make connections between areas of mathematics (through problem-solving). The contexts within which such activity takes place often offer an element of intrigue and can include "real world" contexts or games that do not "dumb down" the mathematics. The contexts or problems themselves should use succinct unambiguous language and offer opportunities for initial success.

Enrichment involves offering opportunities to observe other people being mathematical or the role of mathematics within cultural settings (e.g. art, history, music…)

Experiences for the learners as they engage in mathematics enrichment

When engaging in enriching mathematical activities, learners are drawn into the mathematics either because of the context or the mathematics that emerges from the problem itself. Contexts may result in learners initially experiencing a sense of slight unease. However, through such experiences, the aim is for learners develop as confident and independent, critical thinkers. Learners should be encouraged to be creative and imaginative in their application of knowledge.

Implications for teachers

Teachers will need to identify resources and contexts that support the needs of the learners and the ordered development of skills by utilising, for example, gradients of similarity and complexity. Teachers also need to create an atmosphere in which they engage in dialogue and other interactions including the use of modelling and metacognition and the use of props or cues, as teaching and learning tools. Their aim is to create a community where learners are involved in developing appropriate language that enhances communication as a vehicle for sharing ideas and individual and communal sense making. The individual learner is valued within the group by encouraging them to be creative, independent thinkers who have time to explore starting points and alternative routes. Different approaches are valued but learners also engage in a critical evaluation of effective and efficient methods.

Such settings have the potential to:

In such settings there is a place for everyone. All learners develop confidence in being mathematical and they can create and apply their knowledge beyond the classroom.

Content

The development of the content aspects of the enrichment framework included findings in three key areas: the purposes of problem-solving, the nature of problem-solving and mathematical thinking, and what constitutes a good problem. These findings are discussed below.

Purposes of problem-solving

I identified four purposes for problem-solving, based on and extending (with the inclusion of a fourth purpose) the work of Stanic and Kilpatrick (1998), Nunokawa (2004) and Wilson *et al* (1993). Problem-solving can be viewed as a generic skill applicable to other subjects as well as mathematics and offering the ability to take a critical view of the world (a utility argument). A particular instance of this is engaging in problem-solving as a fundamental part of mathematics, that is problem-solving as a means of being mathematical. This is described as problem-solving for the purposes of being mathematical. Problems associated with this purpose often involve drawing together several mathematical concepts or techniques in order to find solution. Problem-solving can also be used as a tool to learn about the processes of problem-solving (the formative argument of Blum and Niss (1991), i.e. teaching about problem-solving. In a similar way problem-solving can be used as a means of learning mathematical content - teaching through problem-solving. Finally, problem-solving can act as a motivational tool – giving relevance to or a purpose for engaging with, other aspects of mathematics.

The identification of these purposes was useful within the study in supporting the analysis, and purpose of, enrichment materials being produced by the team. This identification also has potential value in further research and practice. For example, different teaching approaches appear to be appropriate when teaching about problem solving compared to situations where you wish to teach through problem solving. Further research is needed to investigate this claim and the benefits to teachers of being aware of these potential differences.

Problem-solving and mathematical thinking

Problem-solving and mathematical thinking strategies were found to be closely associated with views of enrichment. While it was possible to identify a range of literature addressing problem-solving and mathematical thinking, the terms have not been clearly articulated and, as such, this posed a problem for any definition of enrichment that referred to them. As a result, and to add clarity to the notion of enrichment, further analysis of the literature and the occurrence of these concepts in the empirical data were undertaken. A distinction was proposed, with problem-solving referring to more generic skills and mathematical thinking referring to the specific techniques that underpin a mathematical problem-solving process.

Problem-solving refers to generic skills and heuristics such as those described by Polya (1957), Mayer (2002), Wilson, Fernandez *et al* (1993) and others. I propose a model for problem-solving (see Figure 2) which attempts to capture the iterative nature of the process and the messiness that is often associated with it. The model has two key features; it identifies elements of the problem-solving process similar to those offered by many other authors in the current literature, but it also suggests that problem solvers revisit aspects of the process as they move through a problem. For example, as a solver applies mathematical knowledge (analysis and synthesis) they should be reflecting on and evaluating their interim results and methods. As a planned solution is executed, solvers will continually analyse and evaluate in order to monitor and refine the process. The model is therefore intended to act as a focus for discourse on problem-solving with learners. The aim is to be able to model and talk about the process as teachers so that learners can develop a shared understanding.

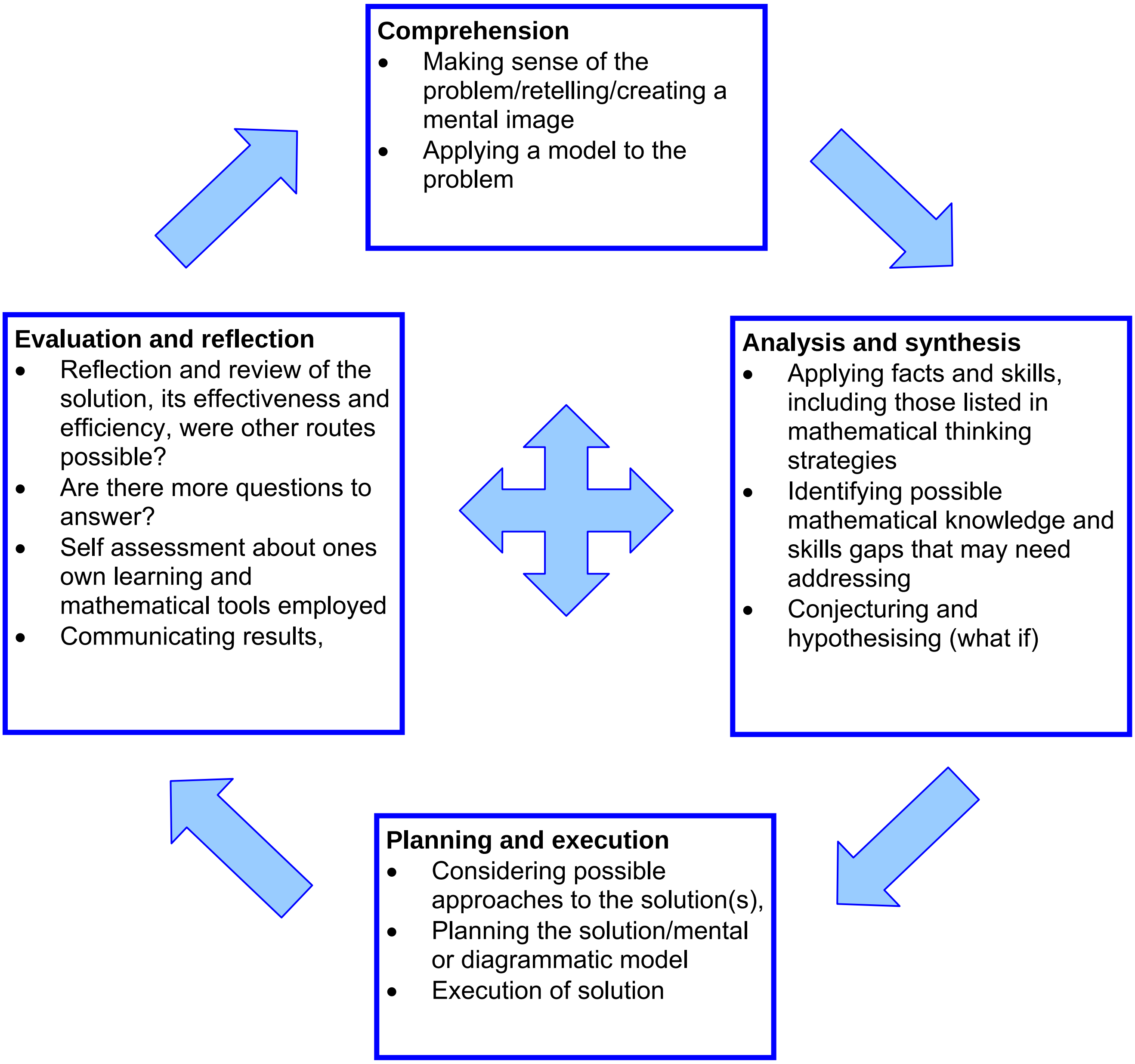


Figure 2: The Problem Solving Model

Mathematical thinking includes specific skills which underpin problem-solving in mathematics. The skills were identified as being of two types: first, those that can be exemplified in specifically mathematical terms such as “being systematic” and second, those aspects that link into the problem-solving heuristics including comprehending the problem and using analysis and synthesis. In addition, mathematical literacy was identified as including the confidence and experience to apply the above skills in unfamiliar contexts.
This aspect of the research has led me to believe that further work on the detailed analysis of these problem-solving and mathematical thinking strategies (including a review of implicit references to them within the existing literature) has the potential to offer three important supporters for teaching and learning. First, such strategies can form the focus of classroom dialogue and offer opportunities for a shared understanding of what is being discussed. Second, students can be encouraged to recognise opportunities to apply these strategies in a range of situations and thereby help to develop their own problem solving skills. Third, they give some clarity in helping teachers, and course designers, to identify what needs to be taught.

Properties of good problems

Through the analysis of empirical data and some implicit commentary within the literature, I was able to identify three properties of “good problems” related to their initial impact, the experience for the solver and the mathematical content of the problem.

The initial impact of the problem is positive if it incorporates a range of key features, which might include the use of succinct clear unambiguous language, a context which draws the solver in, a sense that solving it feels worthwhile, giving opportunities for initial success but with scope to extend and challenge (‘low threshold-high ceiling’ problems).

With a good problem, solvers can be encouraged to think for themselves, often starting their journey with a sense of slight unease. The unease results from the nature of a problem (you do not immediately know the solution or how to find the solution). As a result, learners start on the problem unclear about whether their approach will lead anywhere. To make sense of the context and find a route, they need to step in and explore. This lack of certainty may encourage them to apply what they know in imaginative ways.

A good problem has content which opens up opportunities for a range of learning experiences. To achieve this it could allow for different methods of solution which in turn offer opportunities to identify elegant or efficient approaches. In working towards a solution solvers working on good problems are given opportunities to reveal patterns in mathematics, make generalisations, identify underlying principles or unexpected results. In addition such problems require a solution that calls for a good understanding of process and/or concept and not merely routine following of a given recipe and draws together different mathematical concepts or branches of mathematics.

Mathematical thinking in the trails

Two of the four trails used in the study were designed to support the development of the mathematical thinking skills of “Being Systematic” (Piggott and Pumfrey, 2006) and “Generalising” (Piggott and Pumfrey, 2005). Examination of the types of problems being used in the systematic trail resulted in the identification of different typographies of being systematic. They were: interpretation, framing and deduction; deduction being further subdivided into stepping up, simple-to-complex and all possibilities.
The ability to unpick a mathematical thinking skill (though not entirely successfully in the case of generalising) leads me to propose that similar analyses of other forms of mathematical thinking are possible. This analysis could aid both the development of appropriate content

material and enable such skills to be made explicit when teaching; thereby supporting the understanding and use of those skills. Concerns necessarily follow with regard to the breaking down of such tasks into tick lists that result in success. It is emphasised that this is not the intention of such analysis, rather that the ability to explain what we are doing enables teaching to be better focussed and understandings more likely to be established.

Teaching and Learning

Descriptions of three theoretical perspectives (traditionalist, reflexive and pragmatic were established as an aid to the analysis of aspects of teaching and learning that could be associated with enrichment and might therefore be evidenced in the trails. The perspectives, whilst useful in creating an analytical framework, offered limited value beyond supporting the analysis itself and helping to identify some of the features of teaching and experiences for learners that are associated with enrichment.
The data analysis gave strong evidence of constructivist views of learning held by participants, with descriptions of teaching reflecting this view of coming to know. What was described involved the construction of shared understandings, related to the learning of mathematical concepts, with individuals having their own understanding of that shared knowledge.

There was some evidence of the chunking of ideas in descriptions and purposes of the trails given by participants and in the trails themselves. The purpose of chunking was to offer structure to teaching and learning, mainly at the level of groups of problems (linked because of their similarity or relationship to an aspect of mathematical content or mathematical thinking). This is in contrast to detailed chunking within a problem that leads learners through in stages and reduces complexity, which is common in what might be described as traditional classrooms.

The concept of communities of practice with learners working collaboratively learning through, about and for problem-solving as a shared, social activity was evidenced and there was some resonance with the notion of teacher as the master, practised in the art of problem-solving, who can model and share in problem-solving experiences with the learner as apprentice.

Strong focuses for the approaches to teaching being presented in the enrichment framework were those that valued the autonomy and identity of learners, and that utilised communal sense making, with the identification of misconceptions, metacognition and appropriate intervention as important teaching and learning tools.

Out of these findings stem issues for the experience of learners and the potential to support learners in gaining confidence and independence.

In general, I would hypothesise that much of what is being described is no more than what many would consider to be good classroom practice that is a valuable experience to all learners. The issue seems to be that it is not common practice in many classrooms despite the fact that such practice would be enriching for all learners.

Conclusion

In this paper I have described a framework for enrichment, properties of and relationships between mathematical thinking, problem-solving and enrichment and what constitutes a good problem.

The formulation of a framework for enrichment develops existing poorly framed ideas into a more coherent view of what the term “mathematics enrichment” might mean and offers a starting point for debate within the wider community. It is suggested that, with the exception of acceleration, aspects of enrichment described elsewhere in the literature, do not contradict the ideas of what many have described as a good mathematical experience for all learners. Thus, implying that enrichment is an inclusive, rather than an exclusive, experience. My justification for maintaining the term “enrichment” in these circumstances is that all learners should have an enriching experience when learning mathematics.

Problem solving has been described as a core activity within enrichment and, in attempting to clarify the terms problem-solving and mathematical thinking, I have made explicit the potentially complex underpinning structures related to these terms.

I have highlighted the difficulties of implementing problem solving approaches to teaching while we lack a detailed understanding of what are problem solving and mathematical thinking. In the longer term, I envisage such clarification having the potential to support the development of a curriculum based on problem-solving heuristics and mathematical thinking skills and, through this, drawing out mathematical patterns and connections between content rather than using a curriculum driven by bite–sized mathematics.

The study has also resulted in a list of criteria which can be used to identify a “good problem”. The value of these criteria is in supporting clarity of purpose when choosing and using problems and in encouraging learners to engage in any problem solving activity.
In addition, research is needed to establish whether the practical application of the framework described here can truly offer an enriching experience to learners and also whether enrichment can be more fully implemented on line, especially if it is possible to close many of the gaps related to appropriate and timely intervention identified in the study.

References

Barbe, W. B. (1960) What is Enrichment? In J.F. Magary and J.R. Eichorn (Eds.), *The Exceptional Child: A Book of Readings.* New York: Holt, Rinehart and Winston Inc: 521-523.

Blum, W. and Niss, M. (1991) Applied Mathematical Problem Solving, Modelling, Applications, and Links to Other Subjects - State, Trends and Issues in Mathematics Instruction. *Educational Studies in Mathematics*, 22, 37-68.

Clendening, C. P. and R. A. Davies (1983) *Challenging the Gifted: A guide for Teachers, Librarians and Students.* New York: R. R. Bowker Company.

DfES (2004) *Making Mathematics Count: The Department for Education and Skills response to Professor Sir Adrian Smith's Inquiry into Post-14 Mathematics Education*. Nottingham: DfES.

DfES (2005) Excellence in Cities, The Standards Site, http://www.standards.dfes.gov.uk/sie/eic/ (accessed 23-7-2007).

Ernest, P. (1991) *The Philosophy of Mathematics Education*. London: Routledge Falmer.

Eyre, D. and T. Marjoram (1990) *Enriching and Extending the National Curriculum*. London: Kogan Page.

Feng, W. Y. (2005) *Conceptions of Enrichment. Paper presented at the Cambridge Symposium on Education Research* (CamERA). University of Cambridge, United Kingdom.

George, W. C., Cohn, S. J., Stanley J. C. (eds). (1979) *Educating the Gifted: Acceleration or enrichment.* Revised and expanded proceedings of the Ninth Annual Hyman Blumberg Symposium on Research in Early Childhood Education. Baltimore and London: The Johns Hopkins University Press.

Greeno, J. G.; Collins, A. M. and Resnick, L. B. (1996) Cognition and Learning. In Berliner, D. C. and Calfee, R. C. (eds) *Handbook of Educational Psychology.* New York: Macmillan: 2, 15-46.

Gross, M. U. M. (1999) *From "the saddest sound" to the D Major chord: The gift of accelerated progression.* Keynote address presented at the 3rd Biennial Australasian International Conference on the Education of Gifted Students, Sunday, 15 August, 1999, Melbourne, Australia.

Keating, D. P. (1979) *Educational Acceleration of Intellectually Talented Youths: Prolonged Discussion by a Varied Group of Professionals - Acceleration: Simplistic Gimmickry. Educating the Gifted: Acceleration and Enrichment.* Revised and Expanded Proceedings of Ninth Annual Hyman Blumberg Symposium on Research in Early Childhood Education, John Hopkins University Press Ltd.

Martinson, R. (1968) *Curriculum Enrichment for the Gifted in the Primary Grade*. New Jersey: Prentice-Hall Inc.

Mayer, R. E. (2002) Mathematical Problem Solving. Mathematical cognition. J. M. Royer. Greenwich, CT, *Information Age Publishing*: 69-92.

Piggott, J. and Pumfrey L. (2005) *Maths Trails: Generalising*. Cambridge: CUP.

Piggott, J. and Pumfrey L. (2006) *Maths Trails: Working Systematically*. Cambridge: CUP.

Polya, G. (1957) *How to Solve it.* Princeton: Princeton University Press.

Renzulli, J. (1977) *The enrichment triad model: A guide for developing defensible programs for the gifted and talented*. Wethersfield Ct: Creative Learning Press.

Sheffield (1999) The development of mathematically promising students in the United States. *Mathematics in School*, 28, 3, 15-18.

Smith, A. (2004) *Making Mathematics Count: The Report of Professor Adrian Smith's Inquiry into Post 14 Mathematics Education*. London: The Stationery Office Ltd.

Stanic, G. and Kilpatrick J. (1988) Historical perspectives in problem solving in the mathematics curriculum. In R. Charles and E. Silver (Eds.). *Teaching and learning mathematical problem solving: Multiple research perspectives*. Reston VA: NCTM: 1-22.

Stanley, J. C. (1979) Identifying and Nurturing the Intellectually Gifted. In W. C. George, S. J. Cohn and J. C. Stanley (eds) *Educating the Gifted: Acceleration and Enrichment*. Revised and Expanded Proceedings of the Ninth Annual Hyman Blumberg Symposium on Research in Early Childhood Education. Baltimore: The Johns Hopkins University Press: 172-182.

UK-Mathematics-Foundation (2000) *Acceleration or Enrichment: Serving the needs of the top 10% in School Mathematics*. UK Mathematics Foundation, School of Mathematics, University of Birmingham.

Wallace, B. (1986) Curriculum enrichment then curriculum extension: differentiated educational development in the context of equal opportunities for all children. *Gifted Education International*, 4, 4-9.

Wilson, J. W. M., Fernandez, L., Hadaway, N. (1993) Mathematical Problem Solving. In Wilson P. S. (ed.) *Research Ideas for the Classroom*: High School Mathematics. New York, Macmillan. 2004: 57-78.

Worcester, D. A. (1979) Enrichment in Educating the Gifted: Acceleration and Enrichment. In George, W. C. Cohn, S. J. and Stanley, J. C. Revised and Expanded Proceedings of the Ninth Annual Hyman Blumberg Symposium on Research in Early Childhood Education. Baltimore and London: The Johns Hopkins University Press: 98-106.