Educate~ Special Issue, December 2009, pp 21-36

Research Paper

Drawing, modelling and gesture in students' solutions to a Cartesian product problem

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Contextualisation

There are many students in mainstream secondary education, who, for a variety of reasons, struggle with school mathematics and perform at a level several years below age-related expectations. Although additional support may be provided for these students, it is often unclear exactly what form this should take. It is, however, clear that individuals have different patterns of strengths and weaknesses, and use a remarkable variety of strategies in their work (Dowker, 2005). The most effective remedial support is generally that which is individually tailored to the student, which requires detailed diagnosis of individual students' understandings (and misunderstandings) (Yeo, 2003). The use of visual representation has long been recognised as an important part of understanding and solving mathematical problems, and students' representational choices on set tasks, particularly of a non-canonic type, can combine to provide rich qualitative information on their mathematical thinking.

Abstract*: This paper presents the visual data generated from a single mathematical task, presented to thirteen students during a larger project researching the representational strategies of students with difficulties in mathematics. Students were aged between eleven and fourteen years, from two London comprehensive schools, placed in 'bottom sets' for Mathematics, and listed on their school's Special Educational Needs register. This particular task was a 'Cartesian product' problem requiring them to combine in pairs the elements of two independent sets, in this case within the scenario of finding the total number of different t-shirt/trousers pairings which could be made by choosing from a given selection of these items. No task-specific materials were provided, but paper, coloured pens and multi-link cubes were available. As expected, the majority of the students found this task considerably challenging, but were able to tackle it with varying levels of teacher support. A wide variety of visuo-spatial and/or kinaesthetic representations were produced, including colourful pictorial depictions of the items, physical models both with and without movement into different configurations, and text-based notations. This paper is descriptive, looking in detail at scans and photographs of students' drawings and models, assessing their effectiveness in helping the respective individuals make sense of the task.*

Introduction

Various researchers have investigated children's nonverbal representation, and particularly visual imagery, as a fundamental part of cognitive representation for mathematical problem solving (eg, Boulton-Lewis, 1998; Greer and Harel, 1998; Presmeg, 1998, 2006; Booth and Thomas, 2000; DeWindt-King and Goldin, 2001; DiSessa, 2002, 2004; van Garderen and Montague, 2003). Strong earlier precedents for this interest include Pólya, whose *How to solve it* (Pólya, 1945) instructed students to "Draw a figure. Introduce suitable notation" (p193 in the 1957 edition), and Bruner, who saw knowledge as being actively constructed in the form of connected mental representations (eg, Bruner 1960, 1967). Bruner additionally proposed that children progress through three modes of cognitive representation: *enactive* (manipulating concrete materials), *iconic* (pictorial representation perhaps involving mental imagery) and lastly *symbolic* (competent use of language and mathematical symbols), a framework which has had enduring appeal. It is perhaps unsurprising that this has often been understood, metaphorically, as a set of stepping-stones across which the child moves, one at a time, to achieve competence; however, the enactive and iconic modes "are not superseded ... but assume new and developing roles" (Daniels and Anghileri, 1995, p 41). Additionally, assuming that children may exploit a variety of mathematical strategies in different (and indeed, similar) situations (Dowker, 2005), it is far from straightforward to analyse students' mathematical thinking from their representations of problems; however, such qualitative assessment is required in order to discover not only where lie the gaps in students' understanding, but how best to fill them.

Cartesian product problems

A *Cartesian product* in mathematics is the complete set of ordered pairs resulting from the combination of two sets of elements. Tasks based on this type of multiplicative structure are usually described as *Cartesian product problems* (Brown, 1981; Anghileri, 1989; Nunes and Bryant, 1996; Mulligan and Mitchelmore, 1997; Verschaffel and Corte, 1997; Yeo, 2003), although there are alternate terms, eg, *product of measures* or *partnering* (Vergnaud *et al*, 1979 and Williams and Moore, 1979, respectively, both in Dickson *et al*, 1988).

One way of framing Cartesian products as mathematical tasks, used by several of the above authors, is to represent the two sets of elements as top- and bottom-half clothing items, with the product set represented by the possible top/bottom combinations. The total number of combinations may thus efficiently be calculated by multiplying the number of tops by the number of bottoms. However, if the child does not realise this, the task may also be completed through listing and counting strategies. Nunes and Bryant (1996) describe such an investigation of children's use of some provided materials to support their mathematical reasoning. They compared two different conditions for students: the provision of a complete set of cardboard clothing items (six shorts and four t-shirts) or a partial set (two shorts and four t-shirts), the idea being that students in the 'subset' group could use their limited materials to "create a model for thinking". Although examples were cited of some children recognising the solution as a simple multiplication, overall the rate of correct responses was only just over half for 'complete set' nine-year-olds, and very low for both 'subset' nine-year olds and eight-year olds in either condition. In fact, problems of this type have been used in various educational research studies into both children's understanding of multiplicative structures and their methods of solving 'story' problems, and the general conclusion has been that they are more difficult than other types of multiplicative problem involving similar numbers (Hervey, 1966 in Anghileri, 1989; Brown, 1981; Nesher, 1988 in Nunes and Bryant, 1996); Verschaffel and Corte, 1997; Mulligan and Mitchelmore, 1997; Yeo, 2003).

Nunes and Bryant's above results appear to provide evidence of the difficulty not only of recognising this problem type as a multiplicative situation, but of completing the task at all, that is, through less efficient listing strategies. However, I consider it possible that the provision of task-specific materials in these studies directed students towards a particular representational strategy which was not necessarily the most useful to them, and that it would be interesting to observe the representational strategies of students who did not receive any model clothes. Assuming that of all children who might be set this task, some will not immediately recognise the multiplicative structure, then the success or otherwise of these individuals depends on their creating a countable representation of the solution set. Thus this task is rich in terms of opportunities to find out: a) what kinds of representations students might choose for this simple, closed but nonstandard problem; b) what kinds of representations seem effective in aiding them towards a correct solution; and c) what, if anything, may be gleaned from the students' representations about their understanding of multiplicative structures.

Participants and procedure

The data discussed in this paper comes from thirteen students aged from eleven to fourteen years, from two Inner London mainstream schools, all of whom were listed on their school's Special Educational Needs (SEN) register as having difficulties in numeracy or mathematics, and had been placed in the respective 'bottom sets'. Students were not selected specifically for participation in the task described here, but for the larger project within which it occurred, for which purpose the teachers of bottom sets in Key Stage 3 had been requested to nominate students who struggled the most with the subject; these students were withdrawn from class for tuition sessions. On each occasion, students were provided with plain paper, coloured felt-tips and a bag of coloured multi-link cubes. I collected all paper they had marked, and whenever possible, photographed configurations made with cubes. I also had an audio recorder constantly running to collect verbal responses occurring concurrently with and in addition to their visuo-spatial responses to tasks. However, at this early stage, students' verbalisations during mathematical tasks were quite minimal.

The Cartesian problem used here was not isolated but one of a series of multiplication- and division-based tasks chosen to provide information on students' pre-tuition understanding of multiplicative structures. Prior tasks included abstract number calculations, and 'story' problems of a style likely to be more familiar, for example, sharing a number of items equally, or working out the number of wheels on a given number of cars. 'Holiday clothes' was the final question and expected, in line with past published findings, to be one of the most difficult; although the students were several years older than those in the studies cited above, who were all within the primary age range, their functioning in school mathematics generally appeared to be (and was in some cases stated by educational psychologists to be) of a kind associated with much younger students. I used Anghileri's (1989) and Nunes and Bryant's (1996) Cartesian product task as a model, with changes in presentation relating to its intended function: it was chosen not simply to find out if students would recognise it as a case of multiplication, or if they could obtain a correct answer, but because of its potential for close observing of individual students' representational strategies when problem-solving. Hence, to each student I gave a verbal explanation of the scenario, wrote the list of items (six t-shirts and four pairs of trousers), and gave two verbal examples of possible outfits/combinations. As the emphasis was on the students' own representations, no specific materials were present other than the felt-tips and cubes that I always provided.

Due to external conditions relating to practices within the participating schools, it should be noted that it was not possible to ensure students experienced identical working conditions throughout the task, although it was identically presented. I met with students at one school individually, and at the other in pairs; time was also limited by school-related external factors such as interruptions and alternations to lesson timings. Support was only given in response to direct requests for help, and kept to the minimum judged necessary for a student to continue working on the task, prompts being regarding either mode/media (eg, "Have you thought about using cubes?"), organisation of combinations (eg, "Have you thought about putting them in a table?") or accuracy (eg, "Are you sure there are none missing?"). Students in the 1:2 tuition condition were seated at a distance such that they could not see each other's working, although may have noticed their choice(s) of representational media.

Results

The observable representational modes used by students were writing, drawing, modelling, and gesture, and the media employed were pens/paper and cubes. Students' own hands, and anything held in them, may also be considered as media within the gestural mode. In addition to *mode* and *media*, other emerging aspects of interest were:

- *Motion*: of the modelled representations - whether they were static or included movement of components;
- *Resemblance*: of the drawn or modelled representations - how much the working representation visually resembled the presented task scenario;
- *Completeness*: of the students who reached a correct solution - whether they required a complete solution set of paired items for the total number to be found, or 'jumped' to the answer before the listing process was complete;
- *Consistency*: whether a single strategy was used from start to finish, or changes occurred.

(Note that in this context, *completeness* and *consistency* are not necessarily positive terms.)

Only one Year 9 student, Wendy, used no external representations at all in her solution of the problem. However, it should be noted that this does not imply immediate understanding or ease of solution; she still took ten minutes to arrive at the correct figure, after several incorrect responses. From her (scant) verbalisations, it seems clear she was making use of internal visual representation, discussion of which is outside the scope of this paper. The other twelve students' representational responses are described below. All names have been changed.

Writing

Jenny (Year 7) worked steadily for around ten minutes without requiring any support. She proceeded in a semi-methodical manner by choosing a colour combination and listing it both ways round, eg, black/blue then blue/black (Figure 1). At first, colours were chosen in no discernible order, then clusters appeared, the listing process changing to become more systematic as she continued. At later stages she regularly checked her list and looked for missing combinations from colour subsets before presenting a complete list and counting the pairs. As well as occasional duplicates, Jenny has made a key error in reversing the colours for all her t-shirt/trousers pairs, when it is not actually possible to do so; however, this could indicate evidence of abstraction, the divorcing of the mathematical aspect of a problem from its original scenario.

Figure 1. Jenny (Year 7)

Tasha (Year 8) sometimes exhibited considerable negative emotion regarding mathematics, stating that she “hated” multiplication and division. However, she had proved quite willing to tackle prior tasks with a multiplicative structure providing they were expressed as ‘story problems’, ie, . in non-mathematical language. She spent around thirty minutes on this task overall, although this contained several lapses of attention. It is surprising that she chose a written strategy (Figure 2) as on other occasions she expressed a strong preference for working with cubes. She requested and received support during the task, of which two instances are key. First, she complained about the amount of writing required by her chosen listing strategy, and I suggested using a table. Second, she also became somewhat frustrated as a result of her frequently asking if her list was “finished” and me replying in the negative; however, when I provided the prompt of asking if she had all outfits that included the blue t-shirt, she quickly completed and checked the solution set in a systematic manner.

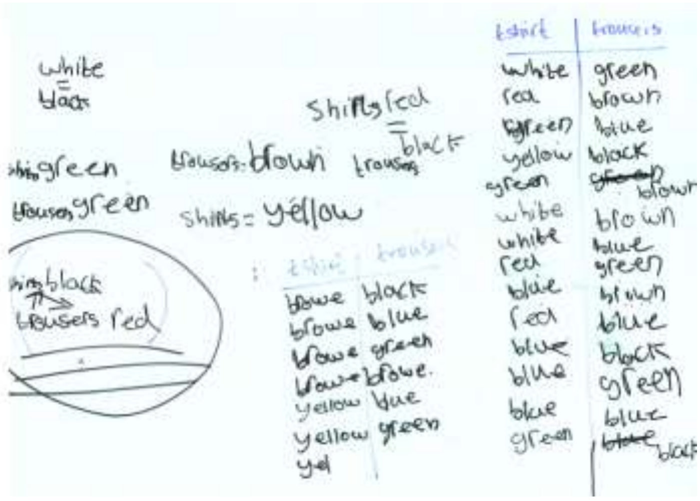

Figure 2. Tasha (Year 8)

Danny (Year 8, paired with **George)** generally tended to rely heavily on counting-based strategies, usually with dot array notation, for multiplicative problems, but used them reliably and with few errors. He spent around five minutes on this task. Danny’s response (Figure 3) differs clearly from the two other written strategies in that it shows an immediate grasp of the problem’s structure, as demonstrated in the orderly working through of groups of combinations. However, two points are of particular interest. First, there is clearly a distinction to be made between (a) comprehending the structure of the complete set of combinations and (b) recognising that the number of combinations may be attained through multiplication. This is perhaps not obvious. Second, Danny's representation is incomplete: he has carefully and consistently listed 20 of the combinations but omitted the last four - yet still giving a correct answer of 24. I suggest that, given his general reliance on visual counting strategies, it was only at this point in the process that he felt confident enough that he could count the last few (unlisted) combinations without the danger of missing out any.

Figure 3. Danny (Year 8)

Drawing

Oscar (Year 9, paired with **Sidney)** was presented with the problem with only five minutes left before the lesson's end, as I considered it possible that he or his co-student might realise that the problem could be solved by multiplication (and so calculate the total within the remaining time). They did not; thus their work is unfinished, and unfortunately neither was there time for development of strategies. Oscar (Figure 4) immediately began to draw one combination after another, without pause. He began with the three same-colour combinations (blue/blue, etc.) There does not appear to be any pattern (yet) in the other combinations listed.

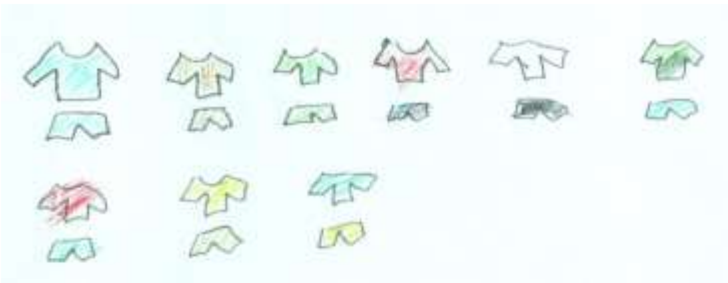

Figure 4. Oscar (Year 9)

Kieran (Year 7, paired with **Harvey)** had solved prior multiplicative problems using counting-based strategies with tally notation. He spent around 35 minutes on the task (although this included several lapses of attention). Kieran chose to draw (Figure 5), barely pausing between the first four combinations. He then made a significant organisational change and reduced the amount of drawing necessary by placing four colours of trousers below each t-shirt. The representation below was presented as Kieran's final solution; it shows 18 outfits which he believed to be the complete solution set. There was not time to check for missing combinations.

Figure 5. Kieran (Year 7)

George (Year 8, paired with **Danny)** had struggled considerably with prior multiplicative problems, often using counting strategies but with no particular preferred or effective notation. However, he needed only around five minutes on this task, my only support was telling him that several of his early estimated or miscounted answers were incorrect. There is a fundamental difference between George's use of drawing (Figure 6) and the previous two (or any of the responses presented so far), in that he does not represent the individual clothing items of the solution set at all, but draws only the relationships between them. As with Danny, this demonstrates a grasp of the problem structure, plus the ability to represent the problem scenario in a more abstract way; however, again, it is important to note that this did not lead him to a multiplication calculation. Also, one might reasonably have expected that, on finding such an elegant representation, it would be trivial to work through it in an orderly manner; however, George's difficulties are apparent, with some links missing or repeated. After one of his suggestions of an incorrect total I informed him there were missing links, and although he looked intently at his representation, he was unable to see where. This suggests perhaps some kind of issue with visual processing.

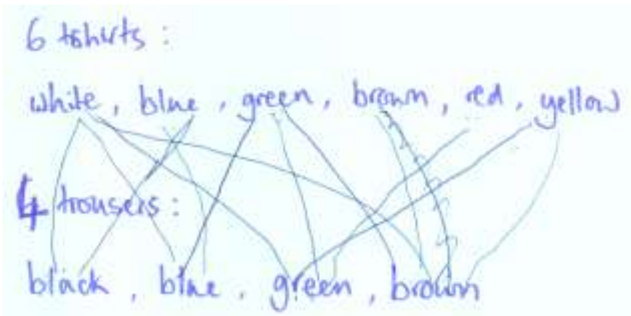

Figure 6. George (Year 8)

Modelling

Three students made significant use of the multilink cubes in representing the problem scenario (although they also used writing to record the results of their modelling).

Sidney (Year 9, paired with **Oscar)**, unfortunately had only a short time on the task. He began by listing two pairs in written format (Figure 7), first just writing unordered pairs of colours, then deciding which were t-shirts and which trousers.

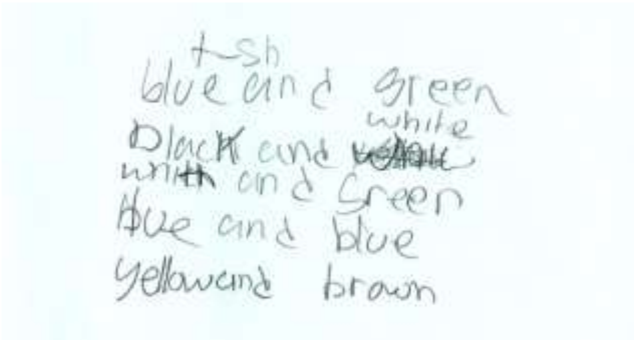

Figure 7. Sidney (Year 9)

However, Sidney was then unable to generate any more. He asked (without prompt) to use cubes, and took pairs of colours then wrote them down (Figure 8). Note that although he made green/brown from cubes, he wrote yellow/brown instead, and so although modelling with cubes would seem to be a promising strategy for him on this task, there are concerns raised regarding the translation of information between the modelled and written modes.

Figure 8. Sydney (Year 9)

Harvey (Year 7, paired with **Kieran)** had struggled to grasp problems involving multiplicative situations, and was highly error-prone even with counting-based strategies, requiring a great deal of teacher support. However, he was highly motivated and could focus for significant periods. Harvey began by listing four pairs of clothing items (Figure 9) before stopping. His first few answers included both valid and non-valid items. He said he could not think of any more, so I suggested cubes might help him (as they had on some prior tasks). He agreed, but required further support. I laid out ten cubes, in two groups corresponding to the six t-shirts and four pairs of trousers, then picked up a black cube, saying "this pair of black trousers, it could go with this one, or this one...". Harvey moved the cube and spoke further combinations. He then picked up and identified another cube as 'blue top' but appeared confused as to what to do with it, so I suggested he go through the different trousers that might go with it. After this, he used this system of picking a t-shirt colour and listing it with each of the four trousers colours to complete the set. At first he was very slow to pick out each new combination, and made some recording errors, but became noticeably quicker as he went on. When he reached the end of his list, he immediately said "Finished!" with unusual confidence, which indicates to me that he was aware the set of possible combinations was now exhausted.

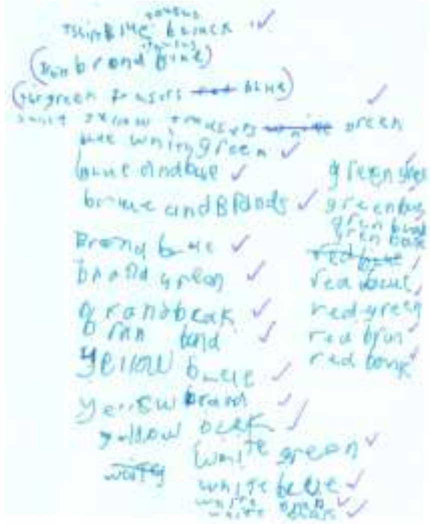

Figure 9. Harvey (Year 7)

Paula (Year 10) was my only student in Key Stage 4 - specifically requested by her school's Head of Mathematics for inclusion. It had been immediately clear that Paula tended to find multiplicative problems extremely difficult, and in fact her addition and even counting were unreliable. However, drawing and (particularly) cubes had enabled her to answer prior number questions that she had been unable to work out in her head. I reduced the numbers in the task to four t-shirts and three trousers, and was prepared to give her a higher level of support if required. We spent around 20 minutes on the task. Paula began by suggesting "[she could wear] black, as there's two blacks". I suggested she make a note of this, in order to keep track of her outfits, but she could or would not make any move to record it. I offered the prompt that she could write her combinations in a table; she concurred this was a good idea, but requested help. I drew one (Figure 10), and demonstrated how she might write combinations. Her next suggestion was "red", so I clarified verbally "a red t-shirt, and what colour trousers?"; she specified black. During this exchange, I took a red cube, then a black cube, and stuck the 't-shirt' and 'trousers' together vertically. I continued to do this as she spoke then wrote combinations.

Figure 10. Paula (Year 10)

As the task progressed, Paula did not seem to have recognised that there would be any pattern governing the list of possible combinations, so was unable to systematically check for 'missing' combinations. My response was to place all the cube pairs in a visuo-spatial sequence, arranged by colour (Figure 11). I left gaps in the appropriate places, explained that some combinations were still to find, and first asked "What should go here?", then made my questioning more explicit, verbally and gesturally, in reference to the visual pattern, ie, "We have the blue t-shirt with the blue trousers, the blue t-shirt with the green trousers; what else must the blue t-shirt go with?" At this point she identified the remaining combinations.

Figure 11. Paula (Year 10)

Gesture

Ellis was the fastest student to complete the task, taking around two minutes. He saw the structure of the problem quickly, but like Danny and George, did not recognise it as being equivalent to calculating 6 x 4. His representation was of the relationships between the two sets of items, but where George drew in the lines connecting them (Figure 6), Ellis simply used a finger to trace them systematically and rhythmically, counting with the fingers of his free hand as he did so.

Mixed-mode responses

All the students above who used cubes also used writing to record the combinations they found, but most if not all of their effective thinking about the problem was done through cube configurations. However, my final two students (paired together) made significant changes of direction in representational strategy, struggling to find the way that would be most effective for them. Both students required a great deal of teacher support, and unfortunately even in a 1:2 situation it was not possible to give each of them the constant attention they needed.

Leo (Year 7) had successfully solved several multiplicative problems, his preferred method being repeated addition. He greatly enjoyed drawing, and tended to produce elaborate pictorial representations for 'story problems'. It is noted that Leo is listed on his school's SEN register as having Asperger Syndrome. Leo began by choosing to draw combinations (Figure 12).

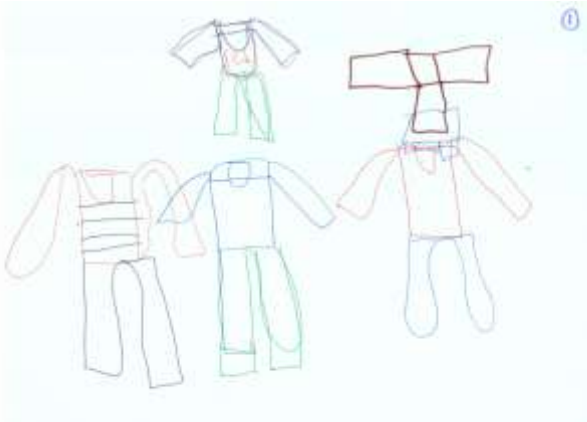

Figure 12. Leo (Year 7)

However, he ran into difficulties because his favourite pen was a four-colour ballpoint. He first wanted to change the question to suit the colours of his pen, rejected my offered pens in the appropriate colours, and chose to switch media to modelling with cubes (Figure 13).

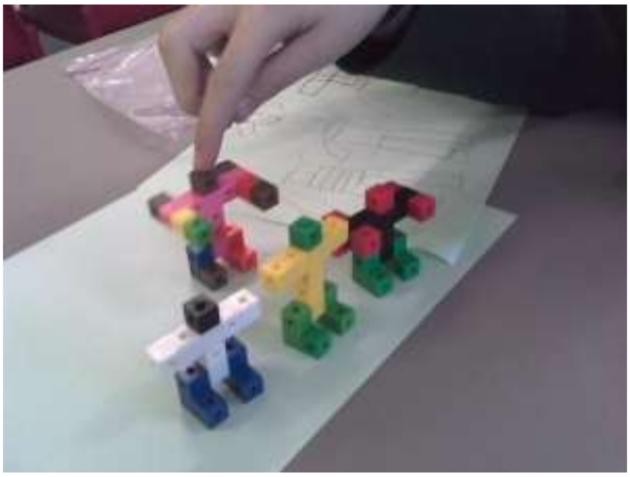

Figure 13. Leo (Year 7)

However, during the making of his (again, elaborate) models, he was taken by the idea that they looked like ‘Transformers’, and started to play with them, after which it was not possible for me to draw him back to the task. Although Leo's response to the task was not greatly helpful in the understanding of his problem-solving strategies or recognition of multiplicative structures, it is interesting that in both modes, he started with valid combinations (ie, from my list of coloured clothing items) that then became increasingly ‘invalid’ ones (eg, yellow/green; white/blue (with a black hat); black/red/ green; many-coloured) as his focus moved away.

Vinny (Year 7) had struggled to grasp prior problems involving multiplicative situations, and was error-prone even with counting-based strategies. He spent around 35 minutes working on the task, requiring a great deal of support. He began with an elaborate drawing, both coloured-in and labelled (Figure 14).

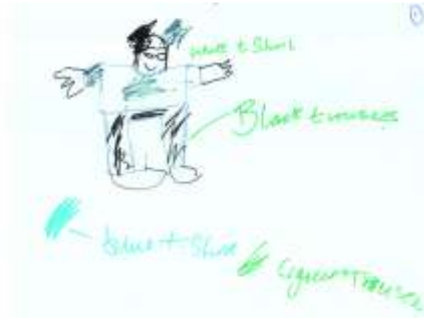

Figure 14. Vinny (Year 7)

He then noticed that Leo had already drawn four figures, and appeared to decide that pictorial representation required too much time and effort. He reduced his drawing to symbolic 'swatches' of colour, but still duplicating the information by writing the names of the items in the appropriate colour (Figure 14 and Figure 15). He then decided that this representational format still required too much effort and changed again, to just writing down the outfits, but, writing being laborious due to weak motor skills, requested my help. I suggested, then drew for him the table, including his outfits so far (Figure 15).

Figure 15. Vinny (Year 7)

He then spoke three more, which I wrote, after which he was willing to take over the (reduced) writing again. As with Jenny and Tasha's lists, it is possible to see the emergence of system, eg, his listing together of all the 'green top' pairs.

Choices

Although the number of participants in this task was small, a rich variety of representations were produced. Six of the students were quick to choose representational *mode* and *media* (Jenny and Danny writing, Kieran, Oscar and George drawing, and Ellis using hand movements) and did not indicate any dissatisfaction with them throughout their time on the task. Of the four students dissatisfied with their initial choice, three quickly switched to modelling with cubes, Sidney and Leo autonomously and Harvey at my suggestion. Vinny retained the media of pen/paper but altered how he used them, trying several differing representational strategies to find the most effective. Paula alone made no real representational choices of her own.

'Drawing' students all drew actual clothing items, ie, representations with a high *resemblance*, although Vinny did briefly simplify his to small blocks of colour. Of the students modelling the problem situation with cubes, Harvey's involved *motion*, in that there were ten cubes, each one representing a single clothing item, which were moved into different configurations, while for the others, each outfit was left in place and new cubes selected for the next. The two schematic representations may be similarly differentiated in terms of motion, in that with George's, the drawn lines provide a static record of all the combinations he had thought of, which could be re-counted and checked, whereas Ellis's finger-movement representation left no permanent trace, so was dependent on accurate and systematic counting.

Effectiveness

To judge the effectiveness of students' representational strategies on this task, it is not enough simply to state what proportion obtained a correct solution, as not only were students provided with support when they requested it, but the majority made incorrect guesses and estimates before arriving at the correct solution, and counting errors occurred even on correct representations. Only two students, Danny and Ellis, gave a correct first answer without support. Two more, Jenny and Kieran, gave almost-correct answers without support (Jenny with duplications of combinations, Kieran with some missing). George, Harvey, Tasha, Vinny and Paula eventually achieved correct solutions with support.

Understanding

It is understood that mathematical understanding is notoriously difficult to assess, but tentative comments may be made on students' understanding of this problem's mathematical structure based on their representations. In particular, levels of *consistency* and *completeness* during the problem-solving process may indicate the presence of any changes in students' thinking regarding the task.

First, although no student calculated the solution via a multiplication, George and Ellis' schematic representations and Danny's systematic list all indicate immediate understanding of the Cartesian structure, ie, that the solution set would be produced by each of the members of the 'tops' set combining with each of the members of the 'bottoms' set. Their use of counting strategies to enumerate the solution set is comparable with their performance on prior and subsequent straightforward multiplicative tasks when, unable to retrieve multiplication facts reliably, they used grouped counting. In particular, George's understanding of the task's structure is shown to be stronger than his ability to execute the necessary procedure. The point at which Danny ceased listing combinations (leaving his representation incomplete) marks not a change in understanding of the structure, but that at which he became confident of counting to a correct total.

Kieran's drawing is unusual for having one clear discontinuity, when he changed from drawing paired outfits to drawing multiple trousers with each t-shirt, which may reflect a new grasp of the structure of combinations. Jenny, Tasha, Harvey and Vinny's lists show a more gradual move to systematicity, with the first combinations being chosen in no particular order, then some grouping of combinations (eg, listing all the blue-trousers combinations together), and eventually using structure to check for missing combinations, ie, that every possible pairing was present.

It is difficult to make judgements on Paula's understanding with any degree of confidence, as she required such a high level of support. However, the fact that she could (a) suggest some combinations, and (b) respond to being presented with a visual sequence of colour combinations and 'fill the gaps' indicates, at least, awareness that elements may be combined, and basic pattern recognition.

Discussion

Choices

Most students' representational choices were unsurprising. However, given their high level of visuo-spatial representational responses (throughout the entire duration of the project) to tasks involving multiplicative structures, and in several cases the great effort it took them to read and write, it is perhaps surprising that writing featured as heavily overall in students' recording processes. On the other hand, it is possible that the high status of writing as a means to communicate one's working and solutions has become ingrained to the extent that many students assume that it is preferred, expected, or even the only kind of working which is acceptable to teachers.

Effectiveness

No mode or media stood out as most successful overall in the attainment of a correct solution; it appears that a variety of representational strategies can be effective, and that (perhaps excepting Paula) students were able to choose one which enabled them to engage with and, given time, to complete the task. Encouragingly, this could be taken as demonstrating a basic level of *metarepresentational competence* (DiSessa, 2002, 2004). However, as all students needed to count the total number of combinations, better organisation of the paired items aided effective and accurate counting, and organisation in turn was dependent on how soon in the problem solving process the Cartesian structure of pairings was perceived.

Understanding

It would be a great oversimplification to categorise students as either understanding the task's mathematical structure or not, particularly if this understanding were defined as recognition of this structure as a case of multiplication. Different levels and different ways of understanding were demonstrated by students at the start, middle and end of the task through the strategies by which they represented the problem, and how they organised the solution sets they produced. However, it is important to remind oneself that the original task as presented did not require students to produce a complete solution set, but to find the number of elements in it. It may be argued that this was the only strategy available to them for finding out the total number, but the fact that during their work on the task several students asked me *how many outfits they had to list/draw/make* is of note; in focusing on their representations of the solution set they had forgotten that enumerating it was the original aim.

Concluding comments

Regarding the historic difficulty of Cartesian product problems, Nesher (1988, in Nunes and Bryant, 1996) pointed out that although they are cases of one-to-many correspondence, this is not explicitly indicated in the verbal formulation, ie, in a 'clothes' task it is up to the problem-solver to figure out the relationship between the numbers of 'tops' and 'bottoms'. Mulligan and Mitchelmore (1997) suggest that fundamental to the effective processing of a multiplicative situation is the recognition of equal-sized groups, and it is that these groups are not at all obvious in the Cartesian situation which causes the particular difficulty. However, I suggest another potential issue lies in the 'temporary' nature of the solution set, namely that in many of the 'Cartesian' scenarios it cannot exist in concrete form. For example, in the 'clothes' version, the set of all possible outfits is countable, of course, as all combinations may be listed, but within the actual problem context, while each pair may be individually constructed, all outfits do not co-exist.

Despite the difficulty, at least some of the students in the cited Cartesian product studies recognised the mathematical situation as an instance of multiplication. Despite generally successful responses to the task, and in some cases quick or immediate appreciation of the structure of the solution set, none of my students multiplied. Can one make any hypotheses as to why this was the case? I suggest a reason may be inadequate and limited mental representations for multiplicative structures. On appraisal of my students' responses to the larger battery of multiplicative tasks, it appears that the majority were aware of multiplication as an operation, with an answer which might be retrieved from memory or found through counting equal groups. Some students were also aware that multiplication is some kind of 'opposite' or 'undoing' operation to division. However, these understandings are both essentially linear and procedural, and lack the concept of multiplication and division as expressing a static two-dimensional relationship between three numbers (eg, from the triplet 3, 6 and 18: 3 x 6 = 18, 6 x 3 = 18, 18/3 = 6 and 18/6 = 3). If so, this may have implications for the way teachers support struggling students in their learning of multiplication and division.

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